RECOVERY OF A TIME-DEPENDENT SOURCE IN A FRACTIONAL LANGEVIN EQUATION

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https://doi.org/10.5281/zenodo.15778222

Abstract: In the current paper, we are interested in studying the time-dependent inverse source problem for the space-degenerate fractional Langevin-type PDE involving a bi-ordinal Hilfer fractional derivative. Sufficient conditions for the given data were established for the existence and uniqueness of the solution. The technique for showing the existence result is based on the uniform convergence of the series.

Keywords: bi-ordinal Hilfer fractional derivative, Fourier-Legendre series, time-dependent inverse source problem

ВОССТАНОВЛЕНИЕ ЗАВИСЯЩЕГО ОТ ВРЕМЕНИ ИСТОЧНИКА В ДРОБНОМ УРАВНЕНИИ ЛАНЖЕВЕНА

Аннотация: В текущей статье мы заинтересованы в изучении зависящей от времени обратной задачи источника для вырожденного по пространству дробного уравнения Ланжевена в частных производных с бипорядковой дробной производной Хильфера. Были установлены достаточные условия для заданных данных для существования и единственности решения. Методика демонстрации результата существования основана на равномерной сходимости ряда.

Ключевые слова: бипорядковая дробная производная Хильфера, ряд Фурье-Лежандра, зависящая от времени обратная задача источника

INTRODUCTION

Studying inverse source problems has been an important target for mathematical research due to its applications in science and engineering [1], [2]. Proving the solution's global (in time) existence and uniqueness is a complicated task. Another interesting thing is that the goal in solving inverse source problems is their solvability and description of a constructive algorithm for finding a solution. We suggest the references [3], [4] for readers to get detailed information about some methods of solving inverse source problems devoted to determining t-dependent factor in the source such as analytical and numerical techniques.

Letting the source term have the form $f(x,t)=a(t)h(x)$, then the inverse the problem consists of determining a source term $a(t)$ and $u(x,t)$ (the temperature distribution in the heat process), from the initial data $\psi(x)$ (initial temperature in heat process) and boundary conditions which come from regularity conditions.

1.1 Statement of the problem

Let $\gamma_1<\gamma_2-\delta_2$, $T>0$ arbitrary fixed time and $\Omega=\{(x,t):-1<x<1,0<t\le T\}$.

The inverse source problem (ISP) here is to find a pair $\{u(x,t),a(t)\}$ functions for given $h(x)$, $\psi(x)$, $\varphi(x)$ such that

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\left(D_{0+}^{(\alpha_2,\beta_2)\mu_2}u(x,t)-\frac{\partial}{\partial x}\left[(1-x^2)u_x(x,t)\right]\right)=h(x)a(t), \tag{1.1}$$

$$\lim_{t\to 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)}u(x,t)=\psi(x), -1\le x\le 1, \tag{1.2}$$

$$\lim_{t\to 0+} I_{0+}^{1-\gamma_1-\delta_2}u(x,t)=\varphi(x), -1\le x\le 1, \tag{1.3}$$

where $h(x)$, $\psi(x)$ and $\varphi(x)$ are given functions, $D_{0+}^{\alpha_i,\beta_i}\mu_i$ is a bi-ordinal Hilfer fractional derivative defined by

$$D_{0t}^{(\alpha_i,\beta_i)\mu_i}y(x):=I_{0+}^{\mu_i(1-\alpha_i)}\frac{d}{dx}I_{0+}^{(1-\mu_i)(1-\beta_i)}y(x), \tag{1.4}$$

where $0<\alpha_i,\beta_i<1$, $0\le\mu_i\le 1$, $i=\overline{1,2}$ and $I_{0+}^{\gamma}y(x)$ is the Riemann-Liouville integral operator of order γ of a function $y(x)$ [6].

We provide the over-determination condition as a way to make the inverse problem uniquely solved:

$$\int_{-1}^{1}u(x,t)dx=E(t), \tag{1.5}$$

where $E(t)\in AC^2([0,T],\mathrm{R})$.

We also consider the following regularity conditions for the solution of the inverse source problem (1.1)-(1.5)

$$t^{1-\delta_2-\gamma_1}u\in C(\overline{\Omega}),\ \ t^{1-\delta_2-\gamma_1}u_x\in C(\overline{\Omega}),\ \ t^{1-\delta_2-\gamma_1}a(t)\in C[0,T],$$

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}D_{0+}^{(\alpha_2,\beta_2)\mu_2}u\in C(\Omega),\ u_{xx}\in C(\Omega).$$

Direct problem related to the ISP was studied in [7] for the equation (1.1) and here we recall some properties of Legendre polynomials which are defined by

$$P_k(x)=\frac{1}{2^k\cdot k!}\frac{d^k(x^2-1)^k}{dx^k}.$$

The Legendre polynomials (see W. Kaplan [8], p. 511) form a complete orthogonal system in $[-1,1]$ and any piece-wise continuous function g can be expressed in the form of Fourier-Legendre series with respect to the system $\{P_k(x)\}$:

$$g(x)=\sum_{k=0}^{\infty}c_kP_k(x), c_k=\frac{(g,P_k)}{\|P_k\|^2}=\frac{2k+1}{2}\int_{-1}^{1}g(x)P_k(x)dx.$$

In this paper, we are also concerned with studying the time-dependent inverse source problem for the equation (1.1) and we take another more favorable condition in order to facilitate calculations instead of a non-local condition.

MAIN RESULTS

Theorem 2.1 *Let* $\gamma_1 < \gamma_2 - \delta_2$, $0 < \left(\int_{-1}^{1} h(x)dx\right)^{-1} < M$ *such that* $h^{(4)}(x) \in L^2(-1,1)$, $\psi''(x) \in L^2(-1,1)$, $\varphi''(x) \in L^2(-1,1)$, $E''(t) \in L^1(0,T)$, *then the unique solution of the inverse source problem (1.1)-(1.5) exists.*

2.1 Uniqueness of the solution

Let there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of the main problem and consider the function $u(x,t) = u_1(x,t) - u_2(x,t)$ which is a solution of the equation (1.1) in the homogeneous case with homogeneous initial conditions

$$\lim_{t\to 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u(x,t) = 0, -1 \le x \le 1, \tag{2.1}$$

$$\lim_{t\to 0+} I_{0+}^{1-\gamma_1-\delta_2} u(x,t) = 0, -1 \le x \le 1, \tag{2.2}$$

Let us consider the following function

$$u_k(t) = \int_{-1}^{1} u(x,t) P_k(x)dx, k = 0,1,2,...,. \tag{2.3}$$

Based on (2.3), we consider the function below

$$v_k(t) = \int_{\varepsilon-1}^{1+\varepsilon} u(x,t) P_k(x)dx, k = 0,1,2,...,. \tag{2.4}$$

where ε is very small positive number.

Applying the operator $D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2}$ with respect to t to both sides of equality (2.4) and using the homogeneous equation corresponding with (??) yield that

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu} v_k(t) = \int_{\varepsilon-1}^{1+\varepsilon} D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2} u(x,t) P_k(x)dx$$

$$= \int_{\varepsilon-1}^{1+\varepsilon} P_k(x) D_{0+}^{(\alpha_1,\beta_1)\mu_1} \frac{\partial}{\partial x}\left[(1-x^2)u_x(x,t)\right]dx,$$

then integrating by parts twice the right side of the last equality and calculating the limit as $\varepsilon \to 0$ give that

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\left[D_{0+}^{(\alpha_2,\beta_2)\mu_2} + \lambda_k\right] u(t) = 0.$$

Obviously, it can be shown that this equation with homogeneous conditions (2.1)-(2.2) has only trivial solution $u_k(t) \equiv 0, t \in [0,T]$ and hence, from (2.3) we get

$$\int_{-1}^{1} u(x,t)P_k(x)dx = 0, k = 0,1,2,...,.$$

Therefore, using the fact of completeness property of system $\{P_k(x)\}$, it is deduced that $u(x,t) \equiv 0$ in Ω, which proves the uniqueness of the solution of ISP. Similarly, the uniqueness result for $a(t)$ can be obtained by doing the steps done above.

2.2 Existence result

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1.1)-(1.5) for arbitrary $a(t) \in C[0,T]$

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)P_k(x), \tag{2.5}$$

where

$$u_k(t) = \psi_k t^{\gamma_2 - 1} E_{\delta_2,\gamma_2}\left(-\lambda_k t^{\delta_2}\right) + \varphi_k t^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}(-\lambda_k t^{\delta_2} +$$

$$+h_k \int_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2,\delta_2+\delta_1}\left[-\lambda_k (t-s)^{\delta_2}\right] a(s)ds, \tag{2.6}$$

with

$$\psi_k = \frac{2k+1}{2}\int_{-1}^{1}\psi(x)P_k(x)dx, \quad \varphi_k = \frac{2k+1}{2}\int_{-1}^{1}\varphi(x)P_k(x)dx,$$

$$h_k = \frac{2k+1}{2}\int_{-1}^{1}h(x)P_k(x)dx, \ k = 0,1,2,...$$

Taking fractional derivative $D_{0+}^{(\alpha_2,\beta_2)\mu_2}$ and $D_{0+}^{(\alpha_1,\beta_1)\mu_1}$ under the integral sign of the over-determination condition (1.5) and in view of the equation (1.1), we obtain

$$a(t) = \left(\int_{-1}^{1} h(x)dx\right)^{-1} \cdot D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2} E(t). \tag{2.7}$$

Substituting (2.7) into (2.6) and assuming that $D_{0+}^{(\alpha_2,\beta_2)\mu_2}E(t) \in L^1(0,T)$, $E(t) \in C[0,T]$ and $E'(t) \in L^1(0,T)$ step by step, we get

$$u_k(t) = \psi_k t^{\gamma_2 - 1} E_{\delta_2,\gamma_2}\left(-\lambda_k t^{\delta_2}\right) + \varphi_k t^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}(-\lambda_k t^{\delta_2} +$$

$$+h_k h^*(E(t) - \lambda_k \int_0^t (t-s)^{\delta_2-1} E_{\delta_2,\delta_2}\left[-\lambda_k (t-s)^{\delta_2}\right] E(s)ds), \tag{2.8}$$

where $h^* = \left(\int_{-1}^{1} h(x)dx\right)^{-1}$.

Now let us establish the necessary condition for $E(t)$ in order to show the existence of (2.7) and here we might take a stronger condition for $E(t)$.

Considering we make sure that $I_{0+}^{1-\gamma_1}\left(D_{0+}^{(\alpha_2,\beta_2)\mu_2}E(t)\right)\in AC[0,T]$ or $D_{0+}^{(\alpha_2,\beta_2)\mu_2}E(t)\in AC[0,T]$. From the last condition we derive that $E'(t)\in C[0,T]$ and $E''(t)\in L^1(0,T)$. These conditions ensure that $a(t)\in C[0,T]$.

From the properties of Legendre polynomials discussed in [7], we recall that

$$|h_k|\le\frac{4\sqrt{2}}{(2k-3)^{3/2}}\|h''(\cdot)\|,\quad |h_k|\le\frac{6\sqrt{2}}{(2k-7)^{7/2}}\|h^{(4)}(\cdot)\|.$$

where $\|\cdot\|$ is a norm of $L^2(-1,1)$.

Now, by taking estimate for the series $t^{1-\delta_2-\gamma_1}u(x,t)$, for all $t\in[0,T]$ we get

$$|t^{1-\delta_2-\gamma_1}u(x,t)|\le\sum_{k=0}^{\infty}|u_k(t)|^2\,|P_k(x)|\le$$

$$\sum_{k=0}^{\infty}|\psi_k|\left|t^{\gamma_2-\delta_2-\gamma_1}E_{\delta_2,\gamma_2}(-\lambda_k t^{\delta_2})\right|^2+\sum_{k=0}^{\infty}|\varphi_k|\left|E_{\delta_2,\delta_2-\gamma_2+\gamma_1+1}(-\lambda_k t^{\delta_2})\right|+$$

$$\sum_{k=0}^{\infty}|h_k||h^*|\left(|E(t)|^2+|E(0)|^2\lambda_k\left|t^{\delta_2}E_{\delta_2,\delta_2+1}(-\lambda_k t^{\delta_2})\right|^2\right)+$$

$$+\sum_{k=0}^{\infty}|h_k||h^*|\lambda_k\int_0^t\left|(t-s)^{\delta_2}E_{\delta_2,\delta_2+1}[-\lambda_k(t-s)^{\delta_2}]\right||E'(s)|ds\le$$

$$\le\sum_{k=0}^{\infty}|\psi_k|\left(\frac{Mt^{\gamma_2-1}}{1+\lambda_k t^{\delta_2}}\right)+\sum_{k=0}^{\infty}|\varphi_k|\left(\frac{Mt^{\delta_2+\gamma_1-1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$+\sum_{k=0}^{\infty}|h_k||h^*|\left(|E(t)|+|E(0)|\lambda_k\left(\frac{Mt^{\delta_2}}{1+\lambda_k t^{\delta_2}})\right)\right)+$$

$$+\sum_{k=0}^{\infty}|h_k||h^*|\int_0^t\left(\frac{M\lambda_k(t-s)^{\delta_2}}{1+\lambda_k(t-s)^{\delta_2}})\right)|E'(s)|ds\le$$

$$\le\|\psi''(\cdot)\|\sum_{k=0}^{\infty}\frac{4\sqrt{2}}{(2k-3)^{3/2}}\left(\frac{Mt^{\gamma_2-1}}{1+\lambda_k t^{\delta_2}}\right)+\|\varphi''(\cdot)\|\sum_{k=0}^{\infty}\frac{4\sqrt{2}}{(2k-3)^{3/2}}\left(\frac{Mt^{\delta_2+\gamma_1-1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$+\|h''(\cdot)\|\sum_{k=0}^{\infty}\frac{|h^*|4\sqrt{2}}{(2k-3)^{3/2}}\left(|E(t)|+|E(0)|\left(\frac{M\lambda_k t^{\delta_2}}{1+\lambda_k t^{\delta_2}})\right)\right)+$$

$$+\|h''(\cdot)\|\sum_{k=0}^{\infty}\frac{|h^*|4\sqrt{2}}{(2k-3)^{3/2}}\int_0^t\left(\frac{M\lambda_k(t-s)^{\delta_2}}{1+\lambda_k(t-s)^{\delta_2}}\right)\left|E'(s)\right|ds.$$

We presume $E'(t)\in C[0,T]$ and considering Weierstrass M- test one can see that the series representation of $t^{1-\delta_2-\gamma_1}u(x,t)$ converges uniformly.

Now, for the second term of the equation (1.1) we have

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}\left[(1-x^2)u_x\right]=-\sum_{k=0}^{\infty}\lambda_k D_{0+}^{(\alpha_1,\beta_1)\mu_1}u_k(t)P_k(x)=$$

$$=-\sum_{k=0}^{\infty}\lambda_k[\psi_k t^{\gamma_2-\delta_1-1}E_{\delta_2,\gamma_2-\delta_1}(-\lambda_k t^{\delta_2})+\varphi_k t^{\delta_2+\gamma_1-\delta_1-1}E_{\delta_2,\delta_2+\gamma_1-\delta_1}(-\lambda_k t^{\delta_2})+$$

$$+h_k h^* D_{0+}^{(\alpha_1,\beta_1)\mu_1}E(t)-\lambda_k h_k h^* E(0)t^{\delta_2-\delta_1}E_{\delta_2,\delta_2-\delta_1+1}(-\lambda_k t^{\delta_2})$$

$$-\lambda_k h_k h^*\int_0^t(t-s)^{\delta_2-\delta_1}E_{\delta_2,\delta_2-\delta_1+1}[-\lambda_k(t-s)^{\delta_2}]E'(s)ds]P_k(x).$$

By taking estimate with help of properties of Legendre polynomials and Mittag-Leffler function, we get

$$\left|D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}\left[(1-x^2)u_x(\cdot,t)\right]\right|\le$$

$$\sum_{k=0}^{\infty}[|\lambda_k\psi_k|\left(\frac{Mt^{\gamma_2-\delta_1-1}}{1+\lambda_k t^{\delta_2}}\right)+|\lambda_k\varphi_k|\left(\frac{Mt^{\delta_2+\gamma_1-\delta_1-1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$+|\lambda_k h_k||h^* D_{0+}^{(\alpha_1,\beta_1)\mu_1}E(t)|+|\lambda_k h_k h^*||\lambda_k E(0)|\left(\frac{Mt^{\delta_2-\delta_1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$+|\lambda_k h_k h^*||\lambda_k E'(0)|\left(\frac{Mt^{\delta_2+1-\delta_1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$+|\lambda_k^2 h_k h^*|\int_0^t\left(\frac{M(t-s)^{\delta_2+1-\delta_1}}{1+\lambda_k(t-s)^{\delta_2}}\right)|E''(s)|ds]\le$$

$$\le\sum_{k=0}^{\infty}\frac{\|\psi''(\cdot)\|4\sqrt{2}}{(2k-3)^{3/2}}\left(\frac{\lambda_k Mt^{\gamma_2-\delta_1-1}}{1+\lambda_k t^{\delta_2}}\right)+\sum_{k=0}^{\infty}\frac{\|\varphi''(\cdot)\|4\sqrt{2}}{(2k-3)^{3/2}}\left(\frac{\lambda_k Mt^{\gamma_2-\delta_1-1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$\sum_{k=0}^{\infty}\frac{6\sqrt{2}\lambda_k\|h^{(4)}(\cdot)\|}{(2k-7)^{7/2}}(|h^* D_{0+}^{(\alpha_1,\beta_1)\mu_1}E(t)|+|E(0)|\left(\frac{M\lambda_k t^{\delta_2-\delta_1}}{1+\lambda_k t^{\delta_2}}\right)+$$

$$\sum_{k=0}^{\infty}\frac{6\sqrt{2}\lambda_k \|h^{(4)}(\cdot)\|}{(2k-7)^{7/2}}\left(|E'(0)|\left(\frac{M\lambda_k t^{\delta_2+1-\delta_1}}{1+\lambda_k t^{\delta_2}}\right)+\right.$$

$$\left.+|h^*|\int_0^t\left(\frac{M\lambda_k (t-s)^{\delta_2+1-\delta_1}}{1+\lambda_k (t-s)^{\delta_2}}\right)|E''(s)|\,ds\right).$$

If $\psi''(x)\in L^2(-1,1)$, $\varphi''(x)\in L^2(-1,1)$, $h^{(4)}(x)\in L^2(-1,1)$ and $E''(t)\in L^1(0,T)$, then the series representation of $\|D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}\left[(1-x^2)u_x\right]$ converges uniformly in Ω. Finally, in a similar way, one can show that the series $D_{0+}^{(\alpha_1,\beta_1)\mu_1}D_{0+}^{(\alpha_2,\beta_2)\mu_2}u$ is uniformly convergent in Ω. This proves the existence of the solution of ISP.

CONCLUSION

In this study, we investigated the inverse problem of determining a time-dependent source function in a space-degenerate fractional Langevin-type partial differential equation involving a bi-ordinal Hilfer fractional derivative. By imposing suitable conditions on the given data, we established the existence and uniqueness of the solution to the inverse problem. The analysis relied on constructing a series representation of the solution and proving its uniform convergence.

The obtained results contribute to the theoretical understanding of inverse problems in fractional models with memory and anomalous diffusion effects, which frequently arise in physics and engineering applications. Future work may involve the development of numerical methods for practical reconstruction of the source term, as well as the extension of the current results to more general forms of fractional operators or multi-dimensional domains.

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